Math Interlude! Neutrino oscillations in matter

Ben Jones, National Nuclear Physics Summer School Lecture 2

1 Two flavor neutrino oscillations in vacuum

Neutrinos oscillate between flavors because they are both massive and mixed. The simplest way to see the emergence of oscillations is to consider the two-flavor approximation, where there are two mass states ν_i for $i = 1, 2$, mixed with two flavors ν_α for $\alpha = e, \mu$. The flavor and mass states are related by a mixing matrix U,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \tag{1}$$

or in matrix notation,

$$|\nu_\alpha\rangle = U_{\alpha i}|\nu_i\rangle. \tag{2}$$

For both the mass states and the flavor states to form complete quantum mechanical bases, U must be unitary,

$$U^\dagger U = 1. \tag{3}$$

And the most general form of a 2x2 unitary matrix is (up to an irrelevant phase factor)

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \equiv \begin{pmatrix} c & s \\ -s & c \end{pmatrix}. \tag{4}$$

Where for shorthand we introduce $c, s = \cos\theta, \sin\theta$. We have introduced the concept of a “mixing angle” θ, which can be anything from 0 to 2π. For two neutrinos, there is only one relevant angle, θ. We consider a state which is born in definite flavor, say the electron flavor,

$$|\psi(t = 0)\rangle = |\nu_e\rangle. \tag{5}$$

Time evolution in quantum mechanics is accomplished via the time evolution operator, annoyingly also called U, but in this lecture we'll just write it in terms of the Hamiltonian $U_{time\,evol} = e^{-iHt}$. We're using natural units, so $\hbar = 1$. Thus,

$$|\psi(t)\rangle = e^{-iHt}|\nu_e\rangle. \tag{6}$$

For vacuum oscillations, the Hamiltonian is diagonal in the mass basis - which can in fact be taken as the definition of the mass basis. We label these energies of the two mass states as E_1, E_2. Thus,

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \tag{7}$$

To apply the operator e^{-iHt} we need to write $|\nu_e\rangle$ in the basis in which e^{-iHt} is diagonal - which is the same basis as that in which H is diagonal. Proceeding to express $|\nu_e\rangle$ in the mass basis, then,

$$|\psi(t)\rangle = ce^{-iE_1t}|\nu_1\rangle + se^{-iE_2t}|\nu_2\rangle. \tag{8}$$

To find the flavor composition at some later time, we project onto a final state flavor state. For example, to find the ν_e survival probability,

$$P_{\nu_e\to\nu_e} = |\langle\nu_e|\psi(t)\rangle|^2 \tag{9}$$

$$= \left|c^2e^{-iE_1t} + s^2e^{-iE_2t}\right|^2 \tag{10}$$

$$= \left(c^2 e^{-iE_1 t} + s^2 e^{-iE_2 t}\right)\left(c^2 e^{-iE_1 t} + s^2 e^{-iE_2 t}\right)^* \tag{11}$$

$$= c^4 + s^4 + 2c^2 s^2 \cos\left[(E_2 - E_1)t\right] \tag{12}$$

Through a little trig magic, we can manipulate this to

$$= c^2(1 - s^2) + s^2(1 - c^2) + 2c^2 s^2 \sin\left[(E_2 - E_1)t\right] \tag{13}$$

$$= 1 - 2\left(s^2 c^2\right)\left(1 - \cos\left[(E_2 - E_1)t\right]\right) \tag{14}$$

$$= 1 - \sin^2 2\theta \sin^2\left[\frac{(E_2 - E_1)t}{2}\right] \tag{15}$$

We note that since only energy **differences** feature in this expression, we can subtract an arbitrary offset energy from all the entries in H without changing the picture. It will be helpful to subtract the energy of a neutrino with zero mass, so

$$H \to \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} - \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix} \tag{16}$$

$$= \begin{pmatrix} \sqrt{p_\nu^2 + m_1^2} - p_\nu & 0 \\ 0 & \sqrt{p_\nu^2 + m_2^2} - p_\nu \end{pmatrix} \sim \begin{pmatrix} m_1^2/2p & 0 \\ 0 & m_2^2/2p \end{pmatrix} \tag{17}$$

And since the neutrino is highly relativistic, for all intents and purposes $p \sim E_\nu$ and $t \sim L$, leading to the oscillation formula

$$P_{e\to e} = 1 - \sin^2 2\theta \sin^2\left[\frac{\Delta m^2 L}{4E}\right] \tag{18}$$

Note that we use an approximation here that is in fact wrong; that all the neutrino mass states have the same momentum. They actually do not - in the decay of a stationary pion, for example, two-body kinematics fixes both the energy and the momentum of each neutrino mass state to distinct values. However, the differences from the above treatment of oscillations are found to be of order Δm^4, and so they do not change the final answer for the oscillation behavior given the small values of Δm^2.

2 Matter Potentials

Neutrinos forward-scattering in matter feel a potential which is non-diagonal in the mass basis, and this complicates their evolution. Neutrinos can forward scatter off of electrons, protons or neutrons in the Earth. The Hamiltonian governing charged current interactions between electrons and neutrinos, for example, takes the following form, of an interaction between currents,

$$H_{CC} = \frac{G_F}{\sqrt{2}}\left[\bar{e}\gamma_\mu(1 - \gamma^5)\nu_e\right]\left[\bar{\nu}_e\gamma^\mu(1 - \gamma^5)e\right]. \tag{19}$$

Through Fierz rearrangement this can be shown to be equal to

$$H_{CC} = \frac{G_F}{\sqrt{2}}\left[\bar{\nu}\gamma_\mu(1 - \gamma^5)\nu\right]\left[\bar{e}\gamma^\mu(1 - \gamma^5)e\right]. \tag{20}$$

The electron is unpolarized, which means only the γ_0 component contributes,

$$\left[\bar{e}\gamma^\mu(1 - \gamma^5)e\right] \sim \bar{e}\gamma^0 e - \bar{e}\gamma^0\gamma^5 e \tag{21}$$

The $e, \nu, \bar{e}, \bar{\nu}$ here spinor wave functions, which can be written in terms of a Weyl spinor ϕ like

$$u(p) = \begin{bmatrix} \phi \\ \frac{\vec{\sigma}.\vec{p}}{E+m}\phi \end{bmatrix}, \tag{22}$$

And for reference, the gamma matrices in the Dirac basis are

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{23}$$

This means that when the electrons are non-relativistic, only the top spinor components are important, killing off the $\gamma^0\gamma^5$ term, leaving us with

$$H = \frac{G_F}{\sqrt{2}} \left[\bar{\nu}(\gamma^0 - \gamma^0\gamma^5)\nu\right] \left[\bar{e}\gamma^0 e\right] = \frac{G_F}{\sqrt{2}} \left[\bar{\nu}_e(\gamma^0 - \gamma^0\gamma^5)\nu\right] \phi^\dagger \phi. \tag{24}$$

$\phi^\dagger\phi$ is nothing other than the electron density, N_e. Thus the charged current contribution to the neutrino-electron matter potential is

$$V^{CC}_{\nu_e,e} = G_F\sqrt{2}N_e. \tag{25}$$

For neutral currents, we get more or less the same thing, but with the relevant g_V and g_A constants inserted,

$$H_{NC} = \frac{G_F}{\sqrt{2}} \left[\bar{\nu}\gamma_\mu(g_V - g_A\gamma^5)\nu\right] \left[\bar{e}\gamma^\mu(g_V - g_A\gamma^5)e\right]. \tag{26}$$

$$g_V = T_3 - 2\sin^2\theta_W Q, \quad g_A = T_3 \tag{27}$$

Adding together the charged and neutral current contributions for each species we obtain the following potentials,

Neutrino flavor	Background flavor	Potential
ν_e	e	$\pm G_F(4\sin^2\theta_W + 1)(N_e + N_{\bar{e}})/\sqrt{2} - \frac{8\sqrt{2}G_F E_\nu}{M_W^2}(\langle E_e\rangle N_e + \langle E_{\bar{e}}\rangle N_{\bar{e}})$
ν_μ, ν_τ	e	$\pm G_F(4\sin^2\theta_W - 1)(N_e - N_{\bar{e}})/\sqrt{2}$
ν_e, ν_μ, ν_τ	n	$\pm G_F(N_{\bar{n}} - N_n)/\sqrt{2}$
ν_e, ν_μ, ν_τ	p	$\pm G_F(1 - 4\sin^2\theta_W)(N_p - N_{\bar{p}})/\sqrt{2}$

Where the $\pm$ in the above expressions refer to neutrinos vs antineutrinos, θ_W is the weak mixing angle, G_F is the Fermi constant, and N_x is the number density of species x. We see that in all cases the matter potential changes sign when switching ν for $\bar{\nu}$. Generally speaking, the average energy of electrons in the material $\langle E_e\rangle \ll m_W$, so only the left term in the top row will matter to us. The second term for $\nu_e e$ scattering that switches on when $E_\nu\langle E_e\rangle \sim M_W^2$ is associated with resonant production of W bosons at very high energies, and we can neglect it in our energy range of interest. Furthermore, we note that in the earth there is very little antimatter, so we assume a background of purely electrons, protons and neutrons, and all the $N_{\bar{x}}$ are zero. As such we are left with the following relevant set of potentials,

Neutrino flavor	Background flavor	Potential in Earth
ν_e	e	$\pm G_F(4\sin^2\theta_W + 1)N_e/\sqrt{2}$
ν_μ, ν_τ	e	$\pm G_F(4\sin^2\theta_W - 1)N_e/\sqrt{2}$
ν_e, ν_μ, ν_τ	n	$\mp G_F N_n/\sqrt{2}$
ν_e, ν_μ, ν_τ	p	$\pm G_F(1 - 4\sin^2\theta_W)N_p/\sqrt{2}$

3 Two-flavor oscillations in matter

The Hamiltonian for our system is now the sum of the vacuum term with the matter potential, as

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + U \begin{pmatrix} V_e & 0 \\ 0 & V_\mu \end{pmatrix} U^\dagger, \tag{28}$$

where

$$V_e = V_{ee} + V_{ep} + V_{en}, \quad V_\mu = V_{\mu e} + V_{\mu p} + V_{\mu n}. \tag{29}$$

This looks set to be a rather complex operator, with all of those contributions to the potential folded in. However, we use again the trick of subtracting a constant energy from the Hamiltonian to simplify it. Subtracting an offset of size V_μ gives us

$$V = U \begin{pmatrix} V_e - V_\mu & 0 \\ 0 & 0 \end{pmatrix} U^\dagger = U \begin{pmatrix} \pm\sqrt{2}G_F N_e & 0 \\ 0 & 0 \end{pmatrix} U^\dagger. \tag{30}$$

We note in passing that we can subtract this constant in any basis, since unitarity of U guarantees that

$$U \begin{pmatrix} V_e & 0 \\ 0 & V_\mu \end{pmatrix} U^\dagger - V_0 = U \begin{pmatrix} V_e - V_0 & 0 \\ 0 & V_\mu - V_0 \end{pmatrix} U^\dagger. \tag{31}$$

Applying the two-flavor PMNS matrix to rotate to the mass basis,

$$U \begin{pmatrix} \pm\sqrt{2}G_F N_e & 0 \\ 0 & 0 \end{pmatrix} U^\dagger = \pm\sqrt{2}G_F N_e \begin{pmatrix} c^2 & -cs \\ -cs & s^2 \end{pmatrix}. \tag{32}$$

And so we have the following total Hamiltonian, in the mass basis,

$$H = \begin{pmatrix} \frac{1}{2E_\nu} m_1^2 \pm \sqrt{2}G_F N_e c^2 & \mp\sqrt{2}G_F N_e cs \\ \mp\sqrt{2}G_F N_e cs & \frac{1}{2E_\nu} m_2^2 \pm \sqrt{2}G_F N_e s^2 \end{pmatrix}. \tag{33}$$

Unless $\cos\theta \sin\theta = 0$, the mass basis is no longer the propagation basis which diagonalizes the Hamiltonian. The neutrino will instead have "effective mass states" ν_{1m}, ν_{2m}, defined by being the basis states that do diagonalize H. In the basis of ν_{1m}, ν_{2m}, the Hamiltonian has the form

$$H_m = \frac{1}{2E_\nu} \begin{pmatrix} M_1^2 & 0 \\ 0 & M_2^2 \end{pmatrix}, \tag{34}$$

Linear algebra tells us that $M_{1,2}^2/2E_\nu$ are simply the Eigenvalues of H, which can be found in any basis. Finding the eigenvalues from Eq. 33,

$$M_{2,1}^2 = \left(m_1^2 + m_2^2 + 2\sqrt{2}G_F N_e E\right) \pm \left[\left(2\sqrt{2}G_F N_e E - \Delta m^2 \cos^2 2\theta\right)^2 + \Delta m^2 \sin^2 2\theta\right]^{1/2}. \tag{35}$$

The new effective ΔM^2 that will appear in oscillations is thus

$$\Delta m_m^2 = 2\sqrt{\left(2\sqrt{2}G_F N_e E - \Delta m^2 \cos^2 2\theta\right)^2 + \Delta m^2 \sin^2 2\theta}. \tag{36}$$

The basis in which H is diagonalized can be represented by a new PMNS-like matrix in matter, which relates the new propagation states to the flavor states,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = U_m \begin{pmatrix} \nu_{1m} \\ \nu_{2m} \end{pmatrix}, \quad U_m = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix}. \tag{37}$$

Giving us an opportunity to define the new effective mixing angles in matter. Working from the eigenvectors of H', we can find that the rotation needed is

$$\sin^2 2\theta_m = \frac{\left(\Delta m^2 \sin 2\theta\right)^2}{\left(2\sqrt{2}G_F N_e E - \Delta m^2 \cos 2\theta\right)^2 + \left(\Delta m^2 \sin 2\theta\right)^2}. \tag{38}$$

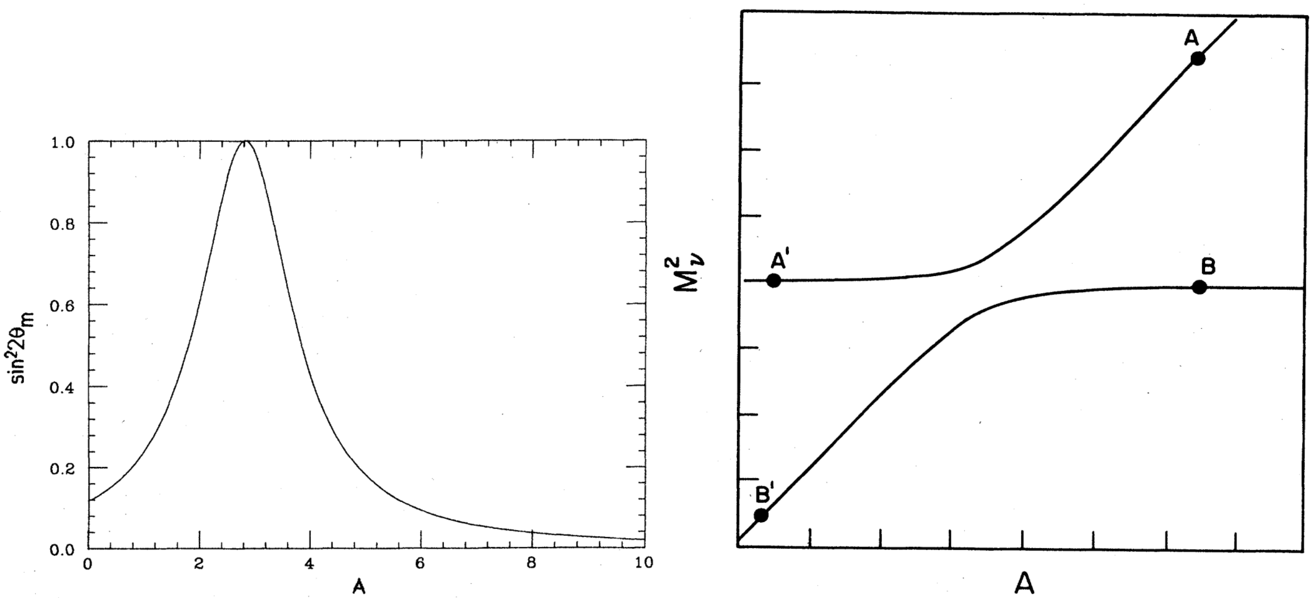
Neutrino oscillations between flavor states in matter will thus be driven by "effective" mass-splitting and mixing angles, as

$$P_{e\to e} = 1 - \sin^2 2\theta_m \sin^2\left[\frac{\Delta m_m^2 L}{4E}\right]. \tag{39}$$

We see that as $N_e \to 0$, both θ_m and Δm_m^2 revert to their vacuum values, as they must. However, we also see that the effective mixing angle can become arbitrarily large, no matter how small the true neutrino mixing angle θ is, for an appropriate density of matter. The density where it really goes off the charts is at

$$N_e^{res} = \frac{\Delta m^2 \cos^2 2\theta}{2\sqrt{2}G_F E}. \tag{40}$$

In matter at that density the mixing becomes maximal, even for small true θ. The plots below shows the dependence of the effective mixing and masses on $A = 2\sqrt{2}G_F N_e E$, showing a broad, "MSW resonant" behavior.



4 Adiabatic transport and solar neutrinos

In realistic applications, the density of matter is not constant, but will vary along the flight path of the neutrino. For the lowest energy neutrinos traveling through the core of the Sun with electron density $N_☼$ we have

$$E \ll \frac{\Delta m^2 \cos^2 2\theta}{2\sqrt{2}G_F N_☼}. \tag{41}$$

In this case,

$$\sin^2 2\theta_m \to \sin^2 2\theta, \quad \Delta m_m^2 \to \Delta m^2 \tag{42}$$

And we recover exactly the same oscillation pattern as neutrinos in vacuum. For the higher energy neutrinos on the other hand,

$$E \gg \frac{\Delta m^2 \cos^2 2\theta}{2\sqrt{2}G_F N_☼}. \tag{43}$$

In the very high energy limit, this drives the effective mixing angle to zero.

$$\sin^2 2\theta_m \sim 0 \tag{44}$$

In practice this means that in middle of the sun, these higher-energy neutrinos are not mixed at all: an electron neutrino is in fact simply a ν_{2m} (in our two flavor picture it would have been a ν_{1m} in the full three-flavor treatment the ordering of the basis states is such that it turns out to be ν_{2m} - but conceptually all the above arguments apply).

Now, we should keep in mind that what a ν_{1m} is actually made of in terms of flavor states is constant changing as the neutrino travels. If the density change is sufficiently slow, however, and the definition of ν_{1m} changes slowly, and the adiabatic theorem of quantum mechanics tells us that the neutrino will adiabatically remain stay a ν_{2m} all along its flight, being born as $\nu_e \sim \nu_{1m}(center)$ and exit the sun as $\nu_1 \sim \nu_{1m}(surface)$. For high energy solar neutrinos, the propagating neutrino changes flavor adiabatically, but it does not oscillate.

As such we make the following predictions about the behavior of solar neutrinos:

1. At low energies, the solar neutrinos experience a vacuum-like propagation, yielding a survival fraction averaged over baselines of

$$P_{\nu_e \to \nu_e}^{low\,E} = 1 - \sin^2 2\theta_{12} \sin^2\left(\frac{\Delta m^2 L}{4E}\right). \tag{45}$$

The neutrinos from the sun are coming from many different baselines, so in fact the oscillation we see is averaged over L, per $\langle\sin^2\left(\frac{\Delta m^2 L}{4E}\right)\rangle = \frac{1}{2}$. The observed oscillation probability for low energy solar neutrinos is thus

$$P^{low\,E}_{\nu_e\to\nu_e} \sim 1 - \frac{1}{2}\sin^2 2\theta_{12}. \tag{46}$$

The best-fit value of θ_{12} today is $\sin^2 2\theta \sim 0.846$ and so we find

$$P^{low\,E}_{\nu_e\to\nu_e} \sim 0.58. \tag{47}$$

2. At high energies, solar neutrinos experience an adiabatic flavor evolution. This means they emerge from the sun as ν_{2m}, and the survival probability is thus

$$P^{high\,E}_{\nu_e\to\nu_e} = |\langle\nu_e|\nu_2\rangle|^2 = \sin^2\theta_{12}. \tag{48}$$

A little trig ID gives us this in terms of $\sin^2 2\theta_{12}$,

$$P^{high\,E}_{\nu_e\to\nu_e} = \frac{1-\cos 2\theta}{2} = \frac{1-\sqrt{1-\sin^2 2\theta}}{2} \sim 0.30. \tag{49}$$

3. In the intermediate region we get a smooth connecting curve. Is is notable that in fact, neither the high or low energy limits actually depend on the neutrino mass differences; as such to measure the solar Δm^2 it is crucial to study this "upturn" region to determine the mass splittings.